Chains of twists for symplectic Lie algebras

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Abstract

Serious difficulties arise in the construction of chains of twists for symplectic simple Lie algebras. Applying the canonical chains of extended twists to deform the Hopf algebras $U(sp(N))$ one is forced to deal only with improper chains (induced by the $U(sl(N))$ subalgebras). In the present paper this problem is solved. For chains of regular injections $U(sp(1)) \subset ... \subset U(sp(N-1)) \subset U(sp(N))$ the sets of maximal extended jordanian twists $\mathcal{F}_{\mathcal{E}_k}$ are considered. We prove that there exists for $U(sp(N))$ the twist $\mathcal{F}_{\mathcal{B}_{0 \prec k}}$ composed of the factors $\mathcal{F}_{\mathcal{E}_k}$. It is demonstrated that the twisting procedure deforms the space of the primitive subalgebra $sp(N-1)$. The recursive algorithm for such deformation is found. This construction generalizes the results obtained for orthogonal classical Lie algebras and demonstrates the universality of primitivization effect for regular chains of subalgebras. For the chain of the maximal length the twists $\mathcal{F}_{\mathcal{B}_{0 \prec k^{\max}}}$ become full, their carriers contain the Borel subalgebra $B^+(sp(N))$. Using such twisting procedures one can obtain the explicit quantizations for a wide class of classical r-matrices. As an example the full chains of extended twists for $U(sp(3))$ is constructed.

1   Introduction

Quantizations of triangular Lie bialgebras with antisymmetric classical r-matrices $r = -r_{21}$ are defined by the twisting elements $\mathcal{F} = \sum f_{(1)} \otimes f_{(2)} \in \mathcal{A} \otimes \mathcal{A}$ that satisfy the twist equations [1]:

$$(\mathcal{F})_{12} \left(\Delta \otimes \mathrm{id} \right) \mathcal{F} = (\mathcal{F})_{23} \left(\mathrm{id} \otimes \Delta \right) \mathcal{F},$$
$$(\epsilon \otimes \mathrm{id}) \, \mathcal{F} = (\mathrm{id} \otimes \epsilon) \, \mathcal{F} = 1. \tag{1}$$

In applications the knowledge of the twisting elements is quite important giving twisted $\mathcal{R}$-matrices $\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21} \mathcal{R} \mathcal{F}^{-1}$ and coproducts

$$\Delta_{\mathcal{F}} = \mathcal{F} \Delta \mathcal{F}^{-1}.$$

Only a few classes of twists can be written explicitly in the closed form [2, 3, 4, 5]. The most interesting of these twists are the so called *extended jordanian twists* $\mathcal{F}_{\mathcal{E}}$ [5] that can be defined on a special type of carrier algebras **L**. These algebras are the analogs of the enlarged Heisenberg algebra and can be found in any simple Lie algebra g of rank greater than 1. The minimal algebra of this type is 4-dimensional:

$$[H, E] = E, \qquad [H, A] = \alpha A,$$
$$[A, B] = E, \qquad [H, B] = \beta B,$$
$$\alpha + \beta = 1.$$

The explicit form of the twisting factors in the extended jordanian twist $\mathcal{F}_{\mathcal{E}}$ is:

$$\Phi_{\mathcal{J}} = e^{H \otimes \sigma},$$
$$\Phi_{\mathcal{E}} = e^{A \otimes B e^{-\beta \sigma}}, \qquad \sigma = \ln \left(1 + E \right).$$

The first factor $\Phi_{\mathcal{J}}$ is the jordanian twist [3] corresponding to the classical r-matrix $r_{\mathcal{J}} = H \wedge E$. The second one, $\Phi_{\mathcal{E}}$, satisfies the twist equations 1 only on the deformed quasiprimitive costructures. In particular the necessary coalgebra can be obtained performing the jordanian twisting by $\Phi_{\mathcal{J}}$ in the initial Heisenberg subalgebra:

$$\Delta_{\mathcal{J}} \left(A \right) = A \otimes e^{\alpha \sigma} + 1 \otimes A,$$
$$\Delta_{\mathcal{J}} \left(B \right) = B \otimes e^{\beta \sigma} + 1 \otimes B,$$
$$\Delta_{\mathcal{J}} \left(E \right) = E \otimes e^{\sigma} + 1 \otimes E.$$

The composition of the twists $\Phi_{\mathcal{J}}$ and $\Phi_{\mathcal{E}}$ defines the extended jordanian twist [5]:

$$\mathcal{F}_{\mathcal{E}} = e^{A \otimes B e^{-\beta\sigma}} e^{H \otimes \sigma}.$$

Here the corresponding classical r-matrix is $r_{\mathcal{E}\mathcal{J}} = H \wedge E + A \wedge B$.

It was demonstrated in [6] that the extended twists $\mathcal{F}_{\mathcal{E}}$ can be composed into *chains*. The latter are based on the sequences of regular injections

$$g_p \subset g_{p-1} \ldots \subset g_1 \subset g_0 = g. \tag{2}$$

To compose the sequence one must choose the *initial* root λ_0^k in each root system $\Lambda\left(g_k\right)$. Let $V_{\lambda_0^k}^{\perp}$ be the subspace orthogonal to the initial root λ_0^k in the system $\Lambda\left(g_k\right)$. The subalgebra g_{k+1} is defined by the subsystem $\Lambda\left(g_{k+1}\right) = \Lambda\left(g_k\right) \cap V_{\lambda_0^k}^{\perp}$. Consider the set π_k of the constituent roots for λ_0^k

$$\begin{aligned}\pi_k &= \left\{ \lambda', \lambda'' \mid \lambda' + \lambda'' = \lambda_0^k; \quad \lambda' + \lambda_0^k, \lambda'' + \lambda_0^k \notin \Lambda\left(g_k\right) \right\} \\ \pi_k &= \pi_k' \cup \pi_k''; \qquad \pi_k' = \left\{ \lambda' \right\}, \pi_k'' = \left\{ \lambda'' \right\}.\end{aligned} \tag{3}$$

It was shown that for the classical Lie algebras one can always find in $g_{\lambda_0^k}^{\perp}$ a subalgebra $g_{k+1} \subseteq g_{\lambda_0^k}^{\perp} \subset g_k$ whose generators become primitive after the twist $\mathcal{F}_{\mathcal{E}_k}$.

Such a process of primitivization of $g_{k+1} \subset g_k$ (called the *matreshka* effect) provides the possibility to compose chains of extended twists of the type

$$\begin{aligned}\mathcal{F}_{\mathcal{B}_{0 \prec p}} &= \prod_{\lambda' \in \pi_p'} \exp\left\{ E_{\lambda'} \otimes E_{\lambda_0^p - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0^p}} \right\} \cdot \exp\{H_{\lambda_0^p} \otimes \sigma_{\lambda_0^p}\} \cdot \\ &\prod_{\lambda' \in \pi_{p-1}'} \exp\left\{ E_{\lambda'} \otimes E_{\lambda_0^{p-1} - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0^{p-1}}} \right\} \cdot \exp\{H_{\lambda_0^{p-1}} \otimes \sigma_{\lambda_0^{p-1}}\} \cdot \\ &\cdots \\ &\prod_{\lambda' \in \pi_0'} \exp\left\{ E_{\lambda'} \otimes E_{\lambda_0^0 - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0^0}} \right\} \cdot \exp\{H_{\lambda_0^0} \otimes \sigma_{\lambda_0^0}\}.\end{aligned} \tag{4}$$

Some peculiarities were found in the case of orthogonal simple algebras [7, 8]. Here the subalgebra $g_{\lambda_0^k}^{\perp}$ is isomorphic to the direct sum $g_{\lambda_0^k}^{\perp} \approx sl(2) \oplus g_{k+1}$. After being twisted by $\mathcal{F}_{\mathcal{E}_k}$ the costructure of the first summand $sl(2)$ is nontrivially deformed while the second $g_{k+1} = so(N - 4(k+1))$ remain primitive. The primitive summands form the carrier space for the canonical chains [6]. These chains are based on the set of injections $so(N - 4(k+1)) \in so(N - 4k)$. In [7] it was proved that each algebra $U_{\mathcal{E}_k}(g_{\lambda_0^k}^{\perp})$ contains

3

not only the deformed subalgebra $U_{\mathcal{E}_k}(sl^k(2))$ but also the primitive Hopf subalgebra $U(sl_G^k(2))$. Due to the fact that the constituent roots (3) form the weight diagram for the representation of $so(N - 4(k + 1))$ in $g_{\lambda_0^k}^\perp$ the invariants of this representation can be used to construct the generators of $sl_G^k(2)$. Thus the universal character of the primitivization effect was confirmed. It was demonstrated that chains of twists for orthogonal algebras can have the properties similar to those of linear simple algebras: after k subsequent steps one can find the primitive subalgebra equivalent to $g_{\lambda_0^k}^\perp$ though it is realized on a *deformed carrier subspace*.

For the symplectic simple Lie algebras the situation is more complicated. The coproducts of generators which correspond to the subalgebra $g_{\lambda_0^k}^\perp \subset g_k$ are nontrivially deformed by the twist $\mathcal{F}_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}$. Hence a proper chain of twists can not be constructed in a canonical way. Nevertheless as we shall demonstrate below there exist in $U_{\mathcal{F}_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}}(g_{k-1})$ the primitive subalgebras equivalent to $g_{\lambda_0^k}^\perp$. The equivalence map is realized by a specific nonlinear transformation of basis. To clarify the algorithm we consider the particular case of $U(sp(3))$. This is the simplest example where the specific structure of symplectic algebras appears.

2 Chains of twists for $sp(3)$

Consider the root system $\Lambda(sp(3))$:

$$\Lambda = \{\pm e_i \pm e_j, \quad \pm 2e_i\} \ (i, j = 1, 2, 3).$$

For the initial root $\lambda_0 = 2e_1$ the constituent roots are $\lambda' = e_1 - e_i$ and $\lambda'' = e_1 + e_i$. We shall use the basis $\left\{ H_{ii}, E_{\lambda_j}, F_{\lambda_j} | \lambda_j \in \Lambda^+(sp(3)) \right\}$. For the nonzero root λ_j let us denote the generators $L_{\lambda_j} = L_{\pm e_i \pm e_j}$ by $L_{\pm i \pm j}$. In this basis the Borel subalgebra $B^+(sp(N))$ is defined by the relations

$$\begin{array}{ll}
[H_{ii}, E_{n+n}] = \delta_{in} E_{n+n}, & [E_{i-j}, E_{n+n}] = 2\delta_{jn} E_{i+n}, \\
[H_{ii}, E_{n+m}] = \frac{1}{2}(\delta_{in} + \delta_{im}) E_{n+m}, & [E_{i-j}, E_{n-m}] = \delta_{jn} E_{i-m} - \delta_{mi} E_{n-j}, \\
[H_{ii}, E_{n-m}] = \frac{1}{2}(\delta_{in} - \delta_{im}) E_{n-m}, & [E_{i-j}, E_{n+m}] = \delta_{jn} E_{i+m} + \delta_{mi} E_{n+j}.
\end{array}$$

The other $sp(N)$-commutators can be obtained using the Chevalley involution $H_{ii} \to -H_{ii}$, $E_{i\pm j} \to F_{i\pm j}$, $F_{i\pm j} \to E_{i\pm j}$.

Here the set of regular subalgebras

$$U(sp(1)) \subset \ldots \subset U(sp(k)) \subset \ldots \subset U(sp(N))$$

4

coincides with the set of subalgebras $g_{\lambda_0^k}^{\perp}$ (compare with (2)). Moreover for the corresponding injections of the root systems

$$\Lambda(sp(1)) \subset \ldots \subset \Lambda(sp(k)) \subset \ldots \subset \Lambda(sp(N))$$

the following property is true: the roots of $\Lambda(sp(k))$ are orthogonal to any long root $\lambda_0 \in \Lambda(sp(N))\backslash\Lambda(sp(k))$. Notice that the chain of regular injections can be based on an arbitrary long root in $\Lambda(sp(N))$. In our particular case the corresponding sets are

$$U(sp(1)) \subset U(sp(2)) \subset U(sp(3)),$$
$$\Lambda(sp(1)) \subset \Lambda(sp(2)) \subset \Lambda(sp(3)).$$

The property mentioned above (for the appropriate ordering of roots) is quite simple: $\Lambda(sp(2)) \perp 2e_1$ and $\Lambda(sp(1)) \perp 2e_1, 2e_2$.

2.1 The first step – the full extended twist

We start the construction of the full chain of twists for $U(sp(3))$ by performing the jordanian twist with the carrier subalgebra generated by $\{H_{11}, E_{1+1}\}$:

$$\Phi_{J_1} = exp\{H_{11} \otimes \sigma_{1+1}\}, \qquad \sigma = \ln(1 + E_{1+1}).$$

The sets $\{2e_1, e_{1-2}, e_{1+2}\}$, $\{2e_1, e_{1-3}, e_{1+3}\}$ define two extensions $\mathcal{E}'$ and $\mathcal{E}''$ for Φ_{J_1}. So the full extended jordanian twist has the twisting element

$$\mathcal{F}_{\mathcal{E}_1} = \underbrace{e^{E_{1-3} \otimes E_{1+3} e^{-\frac{1}{2}\sigma_{1+1}}}}_{\mathcal{E}''} \underbrace{e^{E_{1-2} \otimes E_{1+2} e^{-\frac{1}{2}\sigma_{1+1}}}}_{\mathcal{E}'} \cdot \underbrace{e^{H_{11} \otimes \sigma_{1+1}}}_{J}. \tag{5}$$

Let us write down the corresponding twisted costructure of the subalgebra $g_{\lambda_0^0}^{\perp} = g_{(2e_1)}^{\perp} = U_{\mathcal{E}_1}(sp(2))$,

$$
\begin{aligned}
\Delta_{\mathcal{E}_1}(E_{2+2}) &= E_{2+2} \otimes 1 + 1 \otimes E_{2+2} + 2E_{1+2} \otimes E_{1+2}e^{-\frac{1}{2}\sigma_{1+1}} + \\
&\quad + E_{1+1} \otimes E_{1+2}^2 e^{-\sigma_{1+1}}, \\
\Delta_{\mathcal{E}_1}(E_{2+3}) &= E_{2+3} \otimes 1 + 1 \otimes E_{2+3} + E_{1+2} \otimes E_{1+3}e^{-\frac{1}{2}\sigma_{1+1}} + \\
&\quad + E_{1+3} \otimes E_{1+2}e^{-\frac{1}{2}\sigma_{1+1}} + E_{1+1} \otimes E_{1+2}E_{1+3}e^{-\sigma_{1+1}}, \\
\Delta_{\mathcal{E}_1}(E_{2-3}) &= E_{2-3} \otimes 1 + 1 \otimes E_{2-3}, \\
\Delta_{\mathcal{E}_1}(E_{3+3}) &= E_{3+3} \otimes 1 + 1 \otimes E_{3+3} + 2E_{1+3} \otimes E_{1+3}e^{-\frac{1}{2}\sigma_{1+1}} + \\
&\quad + E_{1+1} \otimes E_{1+3}^2 e^{-\sigma_{1+1}}, \\
\Delta_{\mathcal{E}_1}(F_{2+2}) &= F_{2+2} \otimes 1 + 1 \otimes F_{2+2} + 2E_{1-2} \otimes E_{1-2}e^{-\frac{1}{2}\sigma_{1+1}} - \\
&\quad - E_{1+2}^2 \otimes E_{1+1}e^{-\sigma_{1+1}}, \\
\Delta_{\mathcal{E}_1}(F_{2+3}) &= F_{2+3} \otimes 1 + 1 \otimes F_{2+3} + E_{1-2} \otimes E_{1-3}e^{-\frac{1}{2}\sigma_{1+1}} + \\
&\quad + E_{1-3} \otimes E_{1-2}e^{-\frac{1}{2}\sigma_{1+1}} - E_{1-2}E_{1-3} \otimes E_{1+1}e^{-\sigma_{1+1}}, \\
\Delta_{\mathcal{E}_1}(F_{2-3}) &= F_{2-3} \otimes 1 + 1 \otimes F_{2-3}, \\
\Delta_{\mathcal{E}_1}(F_{3+3}) &= F_{3+3} \otimes 1 + 1 \otimes F_{3+3} + 2E_{1-3} \otimes E_{1-3}e^{-\frac{1}{2}\sigma_{1+1}} - \\
&\quad - E_{1+3}^2 \otimes E_{1+1}e^{-\sigma_{1+1}}.
\end{aligned}
\tag{6}
$$

To continue the construction of the chain we need a primitive subalgebra. The generators E_{2-3}, F_{2-3} (and the corresponding Cartan element) are primitive but these are not sufficient for a proper full chain [6]. It was demonstrated in [9] that the *deformed carrier subspaces*, algebraically equivalent to $U(g_{\lambda_0}^{\perp})$, can be found in $U_{\mathcal{E}}$. Applying the algorithms elaborated in [9] to the Hopf subalgebra $U_{\mathcal{E}}(sp(2))$ with the costructure (6) we find the following basis transformation.

$$
\begin{aligned}
E'_{i+i} &= E_{i+i} - E_{1+i}^2 e^{-\sigma_{1+1}}, \\
F'_{i+i} &= F_{i+i} - E_{1-i}^2, \\
E'_{2+3} &= E_{2+3} - E_{1+2}E_{1+3}e^{-\sigma_{1+1}}, \qquad i = 2, 3 \\
F'_{2+3} &= F_{2+3} - E_{1-2}E_{1-3}, \\
H'_{ii} &= H_{ii}.
\end{aligned}
$$

It can be checked explicitly that these new basic elements are primitive with respect to $\Delta_{\mathcal{E}_1}$. They generate the subalgebra $sp'(2)$.

2.2 The next step – the second extended twist

Now we can temporary forget about the twist (5). Consider the algebra $U(sp'(2))$ on the deformed carrier space generated by E', F' and H'. Intro-

duce for it an independent root system $\Lambda(sp'(2))$. In what follows we shall drop the primes in the notation for the corresponding roots.

Let us perform in $sp'(2)$ the jordanian twist based on the long root $\lambda_{2+2} = 2e_2 \in \Lambda(sp'(2))$

$$\Phi_{J_2} = \exp\{H_{22} \otimes \sigma'_{2+2}\},$$
$$\sigma'_{2+2} = \ln(1 + E'_{2+2}). \tag{7}$$

The set $\{2e_{2+2}, e_{2-3}, e_{2+3}\}$ indicates that there exists the extension $\mathcal{E}_2$ for the twist (7):

$$\mathcal{F}_{\mathcal{E}_2} = \underbrace{e^{E'_{2-3} \otimes E'_{2+3} e^{-\frac{1}{2}\sigma'_{2+2}}}}_{\mathcal{E}} e^{H_{22} \otimes \sigma'_{2+2}}$$

Applying this twist to $U(sp'(2))$ we see that the subalgebra $sp'(1) \in sp'(2)$ becomes nontrivially deformed. Again, the deformed carrier space can be found. The following basis transformation gives the appropriate primitive subalgebra $sp''(1) \in U_{\mathcal{E}_2\mathcal{E}_1}(sp(3))$:

$$\begin{aligned} E''_{3+3} &= E'_{3+3} - (E'_{2+3})^2 e^{-\sigma'_{2+2}}, \\ F''_{3+3} &= F'_{3+3} - (E'_{2-3})^2, \\ H''_{33} &= H_{33}. \end{aligned}$$

2.3 The last step – the jordanian twist

Now consider the subalgebra $U(sp''(1))$ on the twice nontrivially deformed space generated by E'', F'' and H'' and make the last step in our chain construction.

Perform in $U(sp''(1))$ the jordanian twist based on the last long root $\lambda_{3+3} = 2e_3$,

$$\Phi_{J_3} = \exp\{H_{33} \otimes \sigma''_{3+3}\},$$

$$\sigma''_{3+3} = \ln(1 + E''_{3+3}).$$

There are no constituent roots for λ_{3+3}. So the last factor of the chain is purely jordanian:

$$\mathcal{F}_{\mathcal{E}_3} = \Phi_{J_3}.$$

Notice, that in terms of the initial basic elements the deformed generator, E'''_{3+3}, looks like:

$$E''_{3+3} = E_{3+3} - E^2_{1+3} e^{-\sigma_{1+1}} - (E_{2+3} - E_{1+2} E_{1+3} e^{-\sigma_{1+1}})^2 e^{-\sigma'_{2+2}}.$$

Even for rather a small chain it appears extremely difficult to find such complicated deformations of the carrier space without the recursive procedure described above.

In this way the step by step transformations allow us to construct the following chain of twists for $U(sp(3))$:

$$
\begin{aligned}
\mathcal{F}_{\mathcal{G}_{0\prec 2}} &= e^{H_{33}\otimes\sigma''_{3+3}}\cdot \\
&\cdot e^{E'_{2-3}\otimes E'_{2+3}e^{-\frac{1}{2}\sigma'_{2+2}}}e^{H_{22}\otimes\sigma'_{2+2}}\cdot \\
&\cdot e^{E_{1-3}\otimes E_{1+3}e^{-\frac{1}{2}\sigma_{1+1}}}e^{E_{1-2}\otimes E_{1+2}e^{-\frac{1}{2}\sigma_{1+1}}}e^{H_{11}\otimes\sigma_{1+1}}.
\end{aligned}
\tag{8}
$$

Actually the set of injections was also deformed:

$$
U_{\mathcal{F}_{\mathcal{E}_3}}(sp(1)) \subset U_{\mathcal{F}_{\mathcal{E}_3}\mathcal{F}_{\mathcal{E}_2}}(sp(2)) \subset U_{\mathcal{F}_{\mathcal{E}_3}\mathcal{F}_{\mathcal{E}_2}\mathcal{F}_{\mathcal{E}_1}}(sp(3)).
$$

Still it is based on the set of symplectic regular subalgebras. Thus the chain (8) is *proper* for the algebra $sp(3)$ (in contrast with the improper chains described in [6]). The carrier subalgebra for (8) is equivalent to $B^+(sp(3))$. Thus this chain of extended twists is *full*.

3 Conclusions

The importance of explicitly defined twisting elements is that the corresponding quantum R-matrices can be calculated in any representation. The general formula for the triangular universal R-matrix in the case of $sp(3)$ algebra is as follows:

$$
\mathcal{R} = (\mathcal{F}_{\mathcal{G}_{0\prec 2}})_{21}(\mathcal{F}_{\mathcal{G}_{0\prec 2}})^{-1}.
\tag{9}
$$

The classical limit can be obtained after the rescaling (and turning ξ to zero)

$$
E_{i+k} \longrightarrow \xi\eta_k E_{i+k}; \quad k \geq i.
$$

And the classical r-matrix has the form

$$
r = \sum_{k=1}^{3}\eta_k(H_{kk}\wedge E_{k+k} + \kappa_k\sum_{i=k+1}^{3}E_{k-i}\wedge E_{k+i}),
$$

where κ_k (the discrete parameter equal to zero or one) indicates whether we perform in the subalgebra the extended jordanian twist or the pure jordanian

twist. Notice, that only two κ_k's can appear in this formula (κ_1 and κ_2). And in the case of a full chain we must keep $\kappa_k = 1$, for any k. Applying the operator $\exp(\alpha \mathrm{ad}(H_{ss} \otimes 1 + 1 \otimes H_{ss}))$ to quantum R-matrix (9) we can change the values of η_s ($\eta_s \to \eta_s(1 + \alpha)$).

These results can be generalized for the case of an arbitrary symplectic simple algebra $sp(N)$. On any step of a chain the transformations of basis leading to the deformed primitive carrier subspaces can be written explicitly. The corresponding twisting element has the following form:

$$
\begin{aligned}
\mathcal{F}_{\mathcal{G}_{0 \prec (N-1)}} \;=\; & e^{H_{NN} \otimes \sigma^{(N-1)}_{N+N}} \cdot \\
& \cdot e^{E^{(N-2)}_{(N-1)-N} \otimes E^{(N-2)}_{(N-1)+N}} e^{-\frac{1}{2}\sigma^{(N-2)}_{(N-1)+(N-1)}} \, e^{H_{(N-1)(N-1)} \otimes \sigma^{(N-2)}_{(N-1)+(N-1)}} \cdot \\
& \vdots \\
& \cdot e^{E_{2-N} \otimes E_{2+N}} e^{-\frac{1}{2}\sigma'_{2+2}} \ldots e^{E_{2-3} \otimes E_{2+3}} e^{-\frac{1}{2}\sigma'_{2+2}} \, e^{H_{22} \otimes \sigma'_{2+2}} \cdot \\
& \cdot e^{E_{1-N} \otimes E_{1+N}} e^{-\frac{1}{2}\sigma_{1+1}} \ldots e^{E_{1-2} \otimes E_{1+2}} e^{-\frac{1}{2}\sigma_{1+1}} \, e^{H_{11} \otimes \sigma_{1+1}}
\end{aligned}
$$

(10)

Here the elements $\sigma^{(L)}_{K+K}$ are the analogs of σ'_{2+2} and σ''_{3+3} introduced in Section 2.

Thus we can operate with the explicit expression for the corresponding quantum R-matrix

$$
\mathcal{R} = (\mathcal{F}_{\mathcal{G}_{0 \prec (N-1)}})_{21} (\mathcal{F}_{\mathcal{G}_{0 \prec (N-1)}})^{-1}. \tag{11}
$$

According to the general properties of chains [6] every step in (10) can bare its own continuous deformation parameter. Thus the R-matrix has a natural multiparametric form. To get the classical r-matrix one must fix the quantization curve that leads to the classical universal enveloping algebra $U(sp(N))$. The general position of the classical limit curve is obtained when all the parameters are proportional to, say, the length of the curve. In this case the most general classical r-matrix (whose quantum analog is (11)) can be written in the following form:

$$
r = \sum_{k=1}^{N} \eta_k \left(H_{kk} \wedge E_{k+k} + \kappa_k \sum_{i=k+1}^{N} E_{k-i} \wedge E_{k+i} \right).
$$

The continuous parameters η_k describe the ratios of the canonical parameters while the discrete parameters κ_k indicate the possibility to switch off the extension factors in the chain.

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